

Programación lineal

PROGRAMACIÓN LINEAL

PROGRAMACIÓN LINEAL se formula siguiendo el planteamiento general:

$$\min_x z$$

← Función objetivo

s.t.

$$h(x) = 0$$

← Restricciones de igualdad

$$g(x) \leq 0$$

← Restricciones de desigualdad

$$x_{\min} \leq x \leq x_{\max}$$

← Límite variables

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La formulación se debe expresar como:

$$\min_x z = c^T x$$

← Función objetivo

s.t.

$$A_h x = b_h$$

← Restricciones de igualdad

$$A_g x \geq b_g$$

← Restricciones de desigualdad

$$x_{\min} \leq x \leq x_{\max}$$

← Límite variables

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Símbolos empleados

a_{ij} = coefficient of variable j (column) in i^{th} constraint equation (row)

A = matrix with elements a_{ij} (termed LHS)

b = vector of constraint RHS values with elements b_i

c_j = objective coefficient for variable j

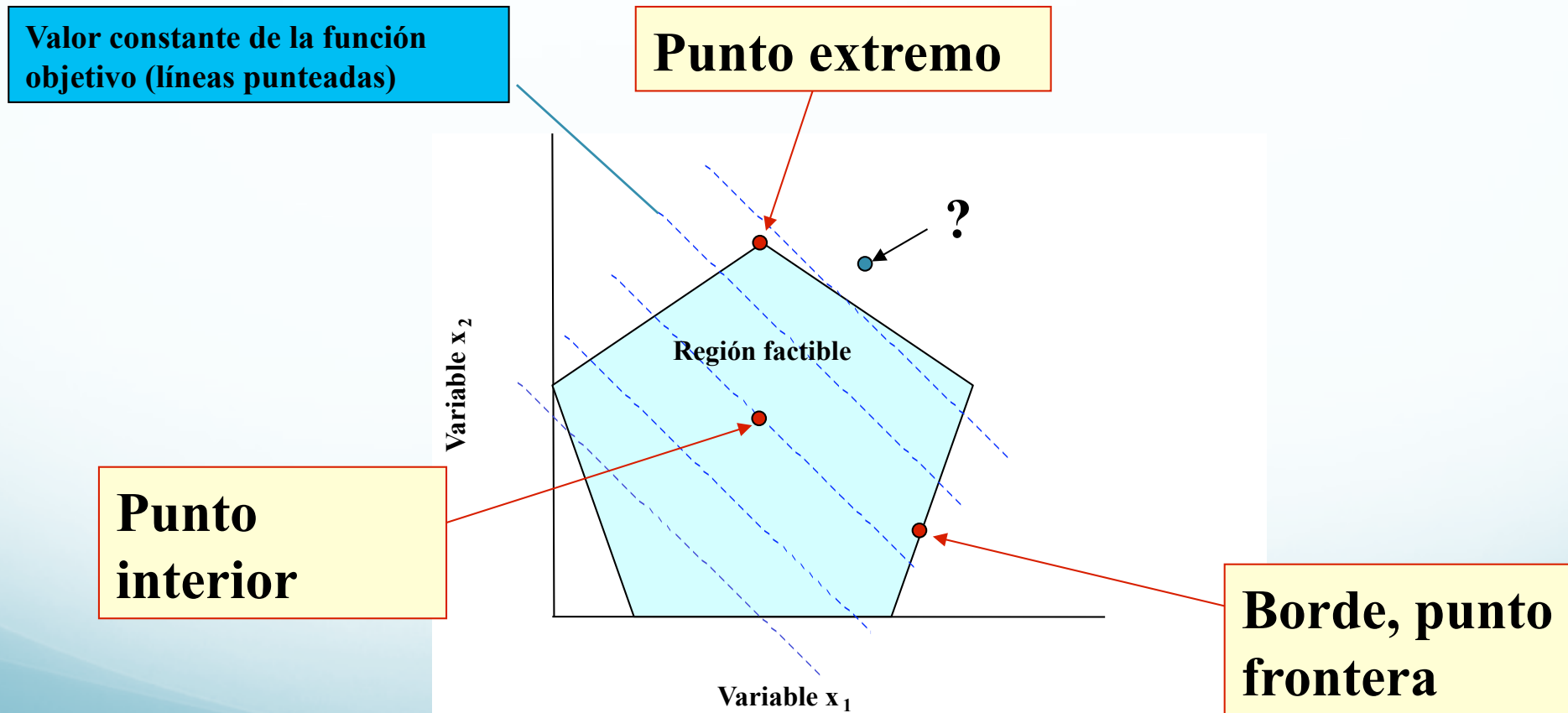
m = number of constraints

n = number of variables

z = Función objetivo = $\sum c_j x_j$

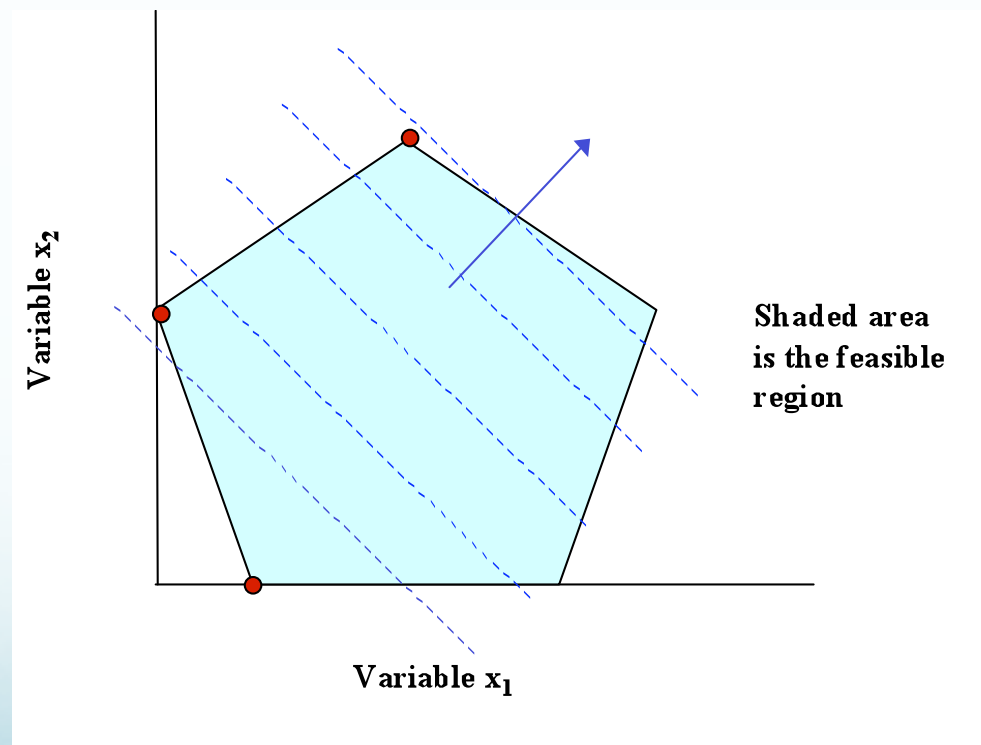
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La región factible tiene tres tipos de puntos.



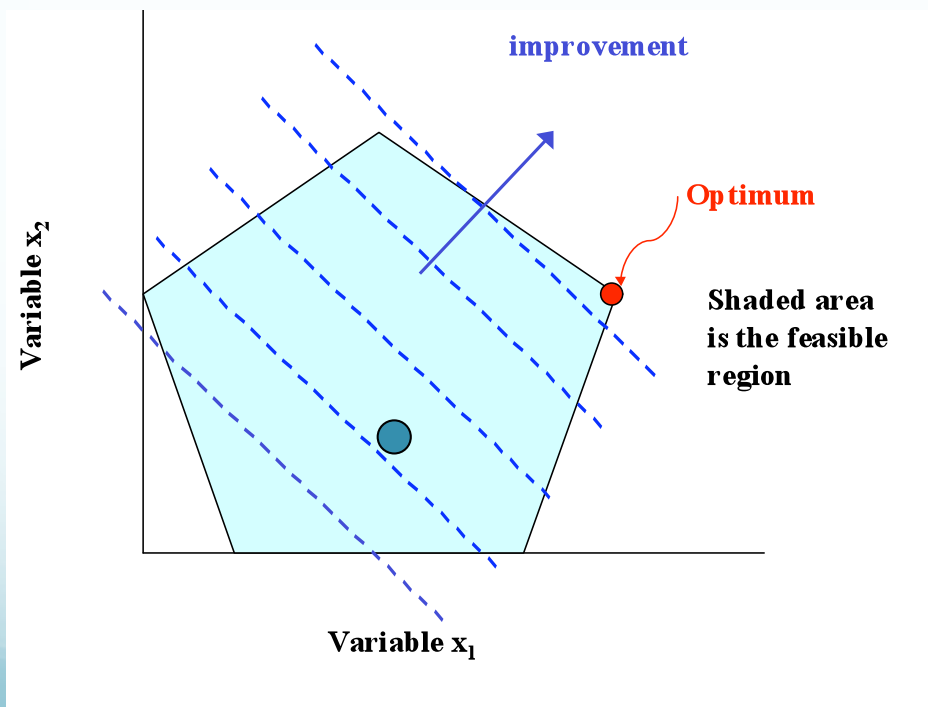
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Punto extremo: Un punto es extremo si cualquier segmento de la región factible que contiene al punto tiene a éste al final del segmento. También conocido como vértice.



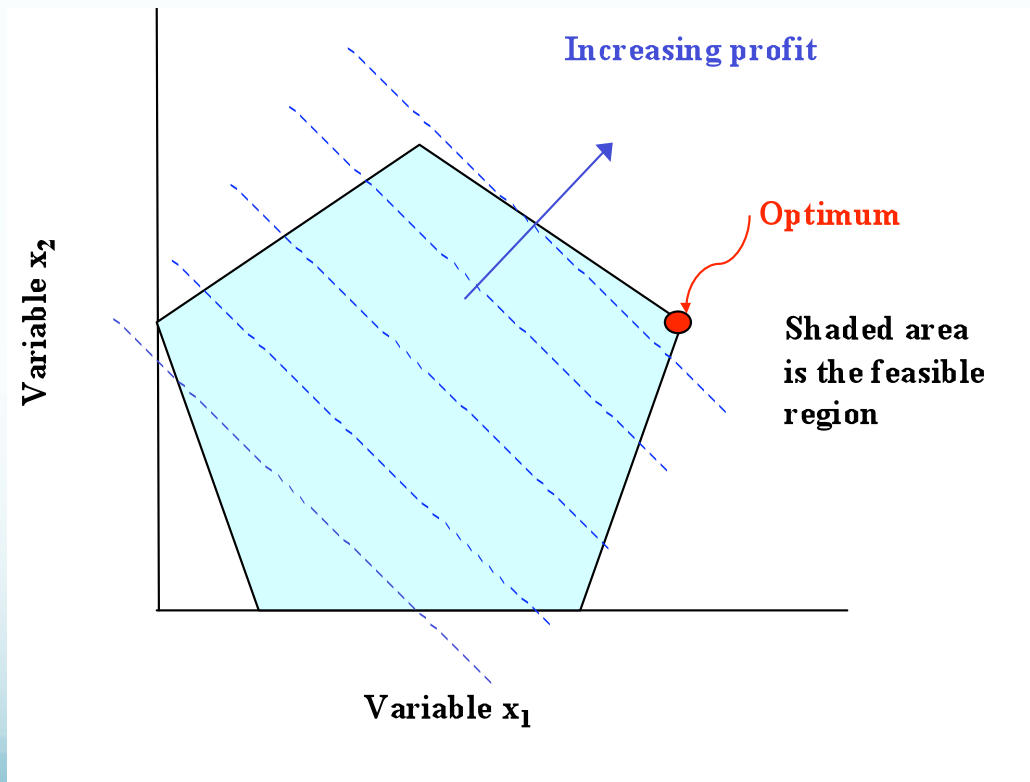
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El mejor vértice debe ser el valor óptimo de la función objetivo! Si el problema tiene solución óptima ésta debe de estar en un vértice; si tiene múltiples soluciones óptimas al menos dos deben de estar en punto extremos.



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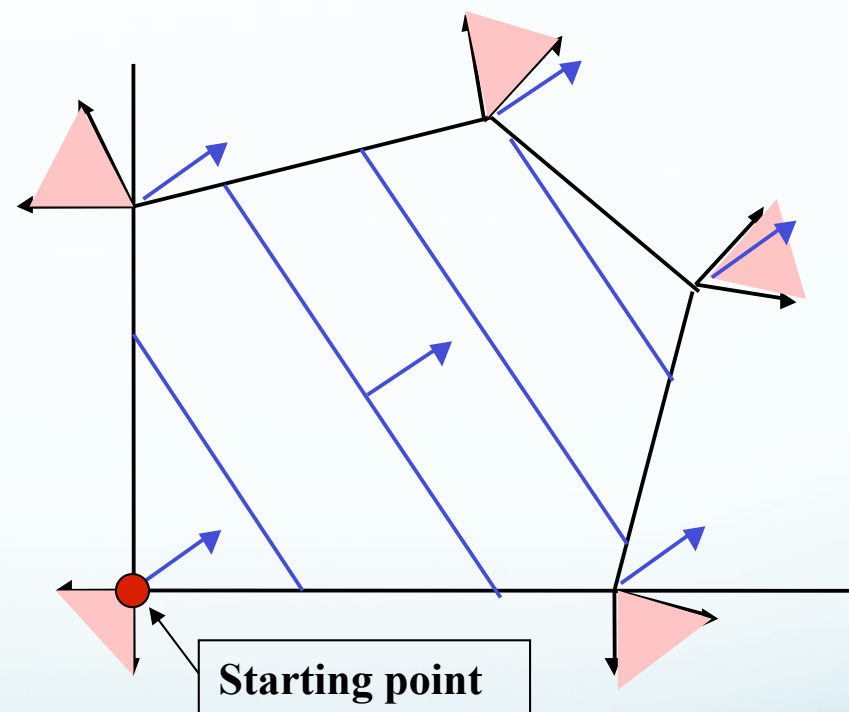
Es un problema de optimización convexa; luego, ¿un óptimo local es un óptimo global!



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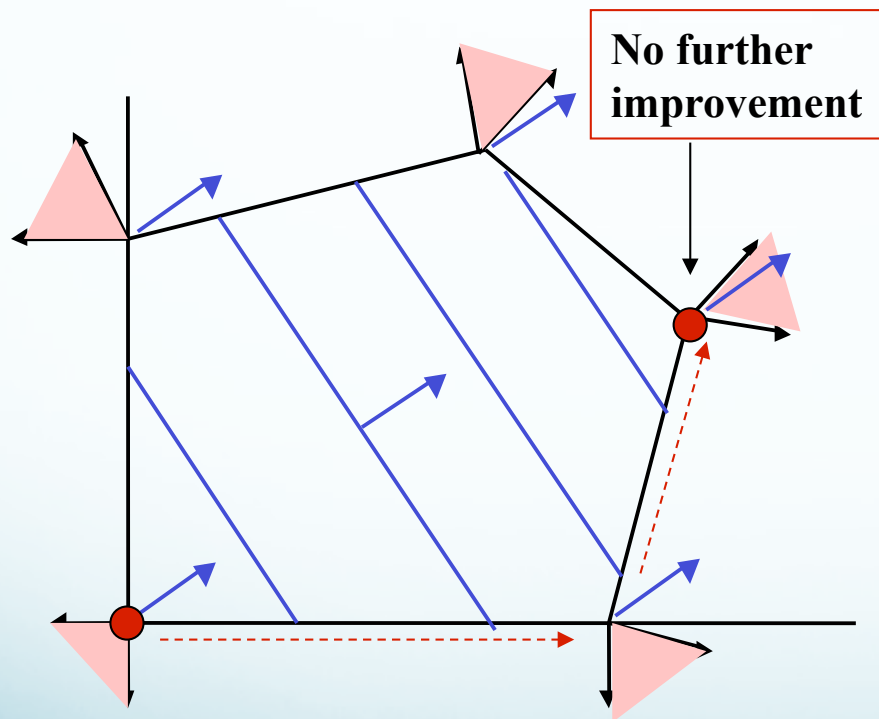
BASIC NUMERICAL OPTIMIZATION

- We only have local information at the “current” point in an iterative scheme.
- We determine a good next point, which will improve the objective value
- We check if further improvement is possible. If yes, continue
- Stop at local optimum.



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Basis of the Simplex Algorithm



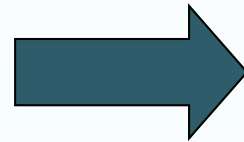
One approach would

- Consider only adjacent corner points for improvement direction.
- Move along the edge that yields the greatest rate of improvement
- Move until another corner point has been reached
- If further improvement is possible, iterate.

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The **Simplex algorithm** - Using the Corner Point concept as the foundation for an efficient solution method.

We want to convert this
general formulation to a
system of linear equations -
we know how to solve these!



$$\min_x z = c^T x$$

s.t.

$$A_h x = b_h$$

$$A_1 x \geq b_1$$

$$A_2 x \leq b_2$$

$$x \geq 0$$

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Standard formulation of LPs

$$\min_x z = c^T x$$

s.t.

$$Ax = b$$

$$x \geq 0$$

How can we formulate the problem...

- If we want to maximize a function?

Use the negative of the same function. *Max f = Min -f*

$$\begin{array}{ll}\min_x & z = c^T x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0\end{array}$$

- If we have unconstrained variables, i.e., x may be negative?

x may be written as the difference of **two non-negative** variables, $x = x_1 - x_2$

- If we have inequalities (besides the equalities)?

We can add **non-negative additional (slack) variables** and convert the inequality into an equality.

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**We will add “slack” variables to all inequalities
to convert them to equalities.**

Original expression

$$5x_1 + 7x_2 - 2.3x_3 \leq 37$$

$$5x_1 + 7x_2 - 2.3x_3 \geq 37$$

$$5x_1 + 7x_2 - 2.3x_3 = 37$$



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The **Simplex algorithm** - Using the Corner Point concept as the foundation for a solution method.

We have the LP problem in “**Standard Form**”. Note that the system has more variables than equations.

$$\min_x z = c^T x$$

s.t.

$$A x = b$$

$$x \geq 0 \quad [\text{note that ultimately, } x_{\min} \leq x \leq x_{\max}]$$

The vector x of variables now includes the slack variables.

Some definitions...

- **Feasible solution.** A solution that satisfies the constraints
- **Basic solution.** A solution in which $n-m$ variables are set equal to zero, and the remaining systems of equations is solved.
- **Basis.** The collection of variables not set equal to zero to obtain the basic solution
- **Basic feasible solution.** This is a basic solution that satisfies the non-negativity conditions.
- **Nondegenerate basic feasible solution.** This is a basic feasible solution that has got exactly m positive x_i .
- **Optimal solution.** A feasible solution that optimizes the Función objetivo.
- **Optimal basic solution.** This is a basic feasible solution for which the Función objetivo is optimal.

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The Simplex Algorithm: Finding a solution for “non-square set of equations

$$\begin{array}{cc}
 \text{Basic} & \text{Non-basic} \\
 \left[\begin{array}{ccc} a_{11} & \dots & a_{1m} \\ \dots & & \\ a_{m1} & & a_{mm} \end{array} \right] & \left[\begin{array}{ccc} a_{1,m+1} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m,m+1} & \dots & a_{mn} \end{array} \right] \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}
 \end{array}$$

Original, non-square equation set of constraints in standard form

$$\left[\begin{array}{ccc} a_{11} & \dots & a_{1m} \\ \dots & & \\ a_{m1} & & a_{mm} \end{array} \right] \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$$

Square set of equations that can be solved for the basic variables

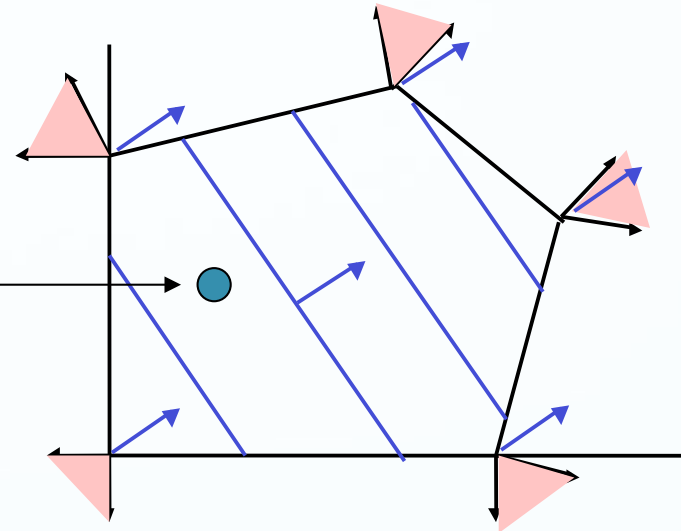
$$\begin{bmatrix} x_{m+1} \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \dots \\ 0 \end{bmatrix}$$

Non-basic variables which take the values that optimize the objective. The values will be at an extreme of allowed range.

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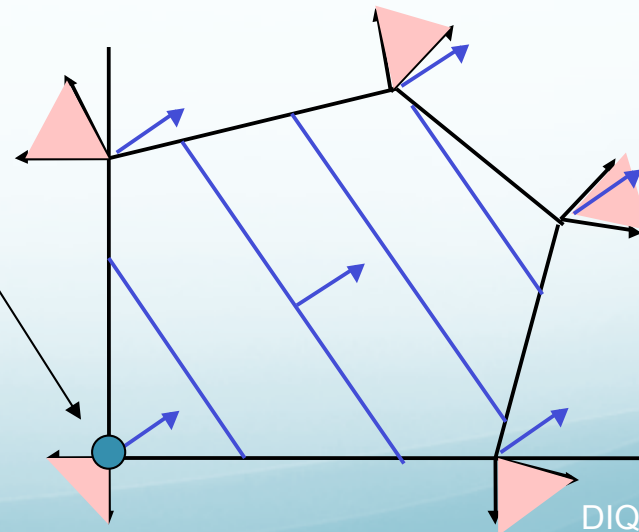
FEASIBLE SOLUTION

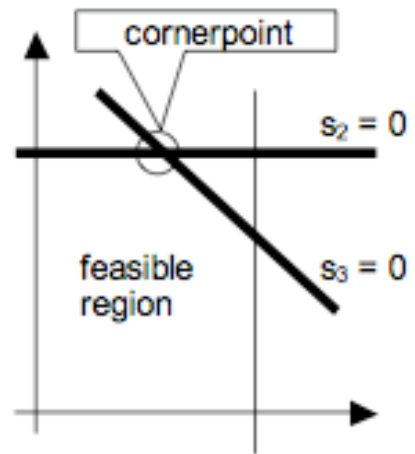
Any selection of values for the non-basic variables that result in all variables being non-negative yields a feasible solution.



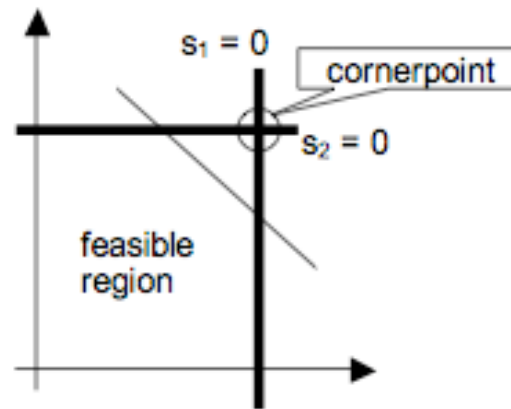
BASIC FEASIBLE SOLUTION

A selection of 0.0 values for the non-basic variables result in basic variables being non-negative and yields a corner point.





(a) feasible cornerpoint



(b) infeasible cornerpoint

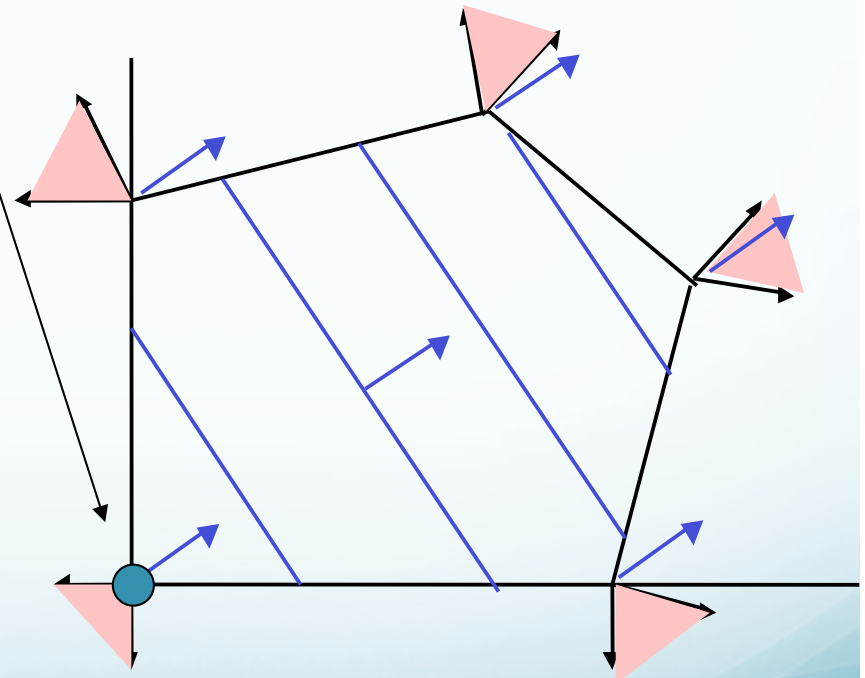
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The Simplex Algorithm

Using the Corner Point concept as the foundation for a solution using the Simplex Algorithm.

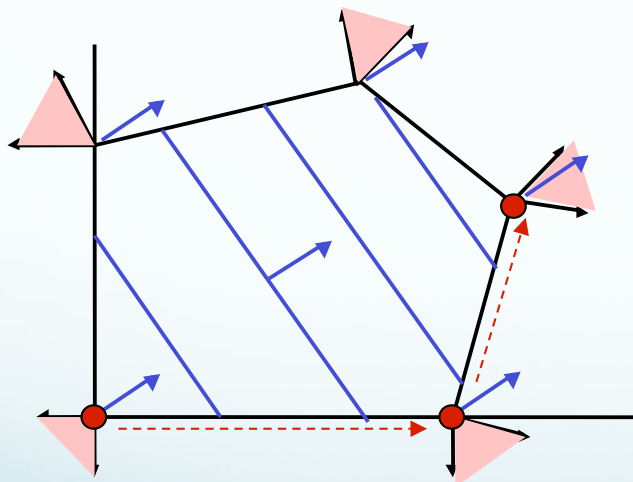
Start here

1. Are we optimal;
2. Move along an edge to an adjacent corner point.
3. This adds one variable to the basis and removes another variable from the basis.



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Moving between adjacent corner points is achieved by changing one variable from basic to non-basic and one non-basic to basic. (The new variables must form a basis.)



The Simplex Algorithm

- Consider only adjacent corner points for improvement direction. ← - - -
- Move along the edge that yields the greatest rate of improvement
- Move until another corner point has been reached
- If further improvement is possible, iterate. - - -

Algoritmo Simplex: Procedimiento.

$$\begin{array}{ll}\text{Mín:} & f(\underline{x}) = -x_1 + x_2 \\ \text{Sujeto a:} & 2x_1 - x_2 \geq -2 \\ & -x_1 + 3x_2 \geq -2 \\ & -x_1 - x_2 \geq -4 \\ & x_1, x_2 \geq 0\end{array}$$

Etapla 0. Transformar las restricciones de desigualdad para que todos los bj sean positivos.

$$\begin{array}{ll}\text{Mín:} & f(\underline{x}) = -x_1 + x_2 \\ \text{Sujeto a:} & -2x_1 + x_2 \leq 2 \quad (\text{A}) \\ & x_1 - 3x_2 \leq 2 \quad (\text{B}) \\ & x_1 + x_2 \leq 4 \quad (\text{C}) \\ & x_1, x_2 \geq 0\end{array}$$

Problema original

Etapla Introducir las variables de holgura (slack variables).

$$\begin{array}{ll}\text{Mín:} & f(\underline{x}) = -x_1 + x_2 \\ \text{Sujeto a:} & -2x_1 + x_2 + x_3 = 2 \quad (\text{A}) \\ & x_1 - 3x_2 + x_4 = 2 \quad (\text{B}) \\ & x_1 + x_2 + x_5 = 4 \quad (\text{C}) \\ & x_1, x_2, x_3, x_4, x_5 \geq 0\end{array}$$

Problema aumentado

¿Grados de libertad?

Etapla 2. Encontrar una solución que esté en un vértice de la región factible.

2 grados de libertad. 2 variables independientes.

2 variables NO BÁSICAS

3 variables dependientes.

3 variables BÁSICAS

$$x_1=0$$

$$x_2=0$$



$$f(\underline{x}) = 0$$

$$x_3 = 2 \quad (A)$$

$$x_4 = 2 \quad (B)$$

$$x_5 = 4 \quad (C)$$

$$\text{Mín:} \quad f(\underline{x}) = -x_1 + x_2$$

$$\text{Sujeto a:} \quad -2x_1 + x_2 + x_3 = 2 \quad (A)$$

$$x_1 - 3x_2 + x_4 = 2 \quad (B)$$

$$x_1 + x_2 + x_5 = 4 \quad (C)$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Etapla 3. Buscar una nueva solución factible que vaya disminuyendo el objetivo.

$$f(\underline{x}) - \sum_i c_i x_i = 0$$

Eliminar como variable no básica la de mayor valor

$$\min \left(\frac{b_j}{a_{jl}} \right) \geq 0$$

Escoger como nueva no básica la de menor valor positivo

$$(A) : 2 / -2 = -1$$

$$(B) : 2 / 1 = 2$$

$$(C) : 4 / 1 = 4$$

(C) x_4 en este caso

Eliminar como variable no básica la de mayor valor -ci

Es la variable que más influye en la función objetivo

Escoger como nueva no básica la de menor valor positivo

Es la variable que llega antes a una restricción.

$$f(\underline{x}) = 0$$

$$x_3 = 2 \quad (A)$$

$$x_4 = 2 \quad (B)$$

$$x_5 = 4 \quad (C)$$

$$\begin{array}{ll} \text{Mín:} & f(\underline{x}) = -x_1 + x_2 \\ \text{Sujeto a:} & -2x_1 + x_2 + x_3 = 2 \quad (A) \\ & x_1 - 3x_2 + x_4 = 2 \quad (B) \\ & x_1 + x_2 + x_5 = 4 \quad (C) \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

$$x_1 = 0$$

$$x_2 = 0$$

$$x_1 - 3 \cdot 0 + x_4 = 2$$



$$x_4 = 2 - x_1$$

Máximo valor de $x_1 = 2$
para que $x_4 \geq 0$

$$x_1 - 0 + x_5 = 4$$



$$x_5 = 4 - x_1$$

Máximo valor de $x_1 = 4$
para que $x_5 \geq 0$

Si se escogiera el mayor valor, entonces x_1 podría llegar a valer hasta 4 con lo cual tendríamos un valor de x_4 negativo.

Etapa 4. Transformar las ecuaciones de acuerdo a la nueva distribución de variables.

Variables básicas: x_1, x_3, x_5

Variables no básicas: x_2, x_4

Restricción más desfavorable

$$(B) : x_1 = 2 + 3x_2 - x_4$$



$$(A): -2(2 + 3x_2 - x_4) + x_2 + x_3 = 2 \Rightarrow x_3 - 5x_2 + 2x_4 = 6$$

$$(B): x_1 - 3x_2 + x_4 = 2$$

$$(C): (2 + 3x_2 - x_4) + x_2 + x_5 = 4 \Rightarrow x_5 + 4x_2 - x_4 = 2$$

$$f + (2 + 3x_2 - x_4) - x_2 = 0 \Rightarrow f + 2x_2 - x_4 = -2$$

Se continua el procedimiento hasta....?

Todos los coeficientes de la función objetivo son negativos.

Procedimiento en forma de tabla

	x_1	x_2	x_3	x_4	x_5	f	b	
x_3	-2	1	1	0	0	0	2	
x_4	[1]	-3	0	1	0	0	2	← fila pivote
x_5	1	1	0	0	1	0	4	
	1	-1	0	0	0	1	0	

columna
pivote

	x_1	x_2	x_3	x_4	x_5	f	b	
x_3	0	-5	1	2	0	0	6	
x_1	1	-3	0	1	0	0	2	
x_5	0	4	0	-1	1	0	2	←
	0	2	0	-1	0	1	-2	

	x_1	x_2	x_3	x_4	x_5	f	b
x_3	0	0	1	0.75	1.25	0	8.5
x_1	1	0	0	0.25	0.75	0	3.5
x_2	0	1	0	-0.25	0.25	0	0.5
	0	0	0	-0.5	-0.5	1	-3

$$\begin{aligned} x_3 &= 8.5 \\ x_1 &= 3.5 \\ x_2 &= 0.5 \end{aligned}$$

Forma canónica

$$\begin{array}{ll}\text{Mín:} & f(\underline{x}) = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{Sujeto a:} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ & x_i \geq 0 ; b_j \geq 0 \quad (i = 1, n ; j = 1, m)\end{array}$$

Cuando la solución básica factible inicial no es trivial:

Método de las dos fases

Método de la gran M

Método de las dos fases

$$\text{Mín:} \quad f = x_1 + 2x_2$$

$$\text{Sujeto a:} \quad 3x_1 + 4x_2 \geq 5$$

$$x_1 + x_2 \leq 4$$

$$\text{Mín:} \quad f = x_1 + 2x_2$$

$$\text{Sujeto a:} \quad 3x_1 + 4x_2 - x_3 = 5 \quad (\text{A})$$

$$x_1 + x_2 + x_4 = 4 \quad (\text{B})$$

$$x_i \geq 0$$

$$w + \left(\sum_{i=1}^m a_{i1} \right) x_1 + \left(\sum_{i=1}^m a_{i2} \right) x_2 + \dots + \left(\sum_{i=1}^m a_{in} \right) x_n = \sum_{i=1}^m b_i$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + x_{n+2} = b_2$$

$$\vdots \quad \ddots \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_{n+m} = b_m$$

Fase I

Mín: $f = x_1 + 2x_2$

Sujeto a: $3x_1 + 4x_2 - x_3 + x_5 = 5$ (A)

$x_1 + x_2 + x_4 + x_6 = 4$ (B)

$x_i \geq 0$

$w + 4x_1 + 5x_2 - x_3 + x_4 = 9$

x_1	x_2	x_3	x_4	x_5	x_6	w	b	
3	4	-1	0	1	0	0	5	←
1	1	0	1	0	1	0	4	
4	5	-1	1	0	0	1	9	

x_1	x_2	x_3	x_4	x_5	x_6	w	b	
0.75	1	-0.25	0	0.25	0	0	1.25	
0.25	0	0.25	1	-0.25	1	0	2.75	←
0.25	0	0.25	1	-1.25	0	1	2.75	

x_1	x_2	x_3	x_4	x_5	x_6	w	b
1	1	0	1	0	1	0	4
1	0	1	4	-1	4	0	11
0	0	0	0	-1	-1	1	0

Fase II

Mín: $f = x_1 + 2x_2$

Sujeto a: $3x_1 + 4x_2 - x_3 = 5$ (A) $x_2 = 4$; $x_3 = 11$.

$x_1 + x_2 + x_4 = 4$ (B)

$x_i \geq 0$

$$f = x_1 + 2(4 - x_1 - x_4) \Rightarrow f + x_1 + 2x_4 = 8$$

x_1	x_2	x_3	x_4	f	b	
1	1	0	1	0	4	
1	0	1	4	0	11	←
1	0	0	2	1	8	
			↑			

x_1	x_2	x_3	x_4	f	b	
0.75	1	-0.25	0	0	1.25	←
0.25	0	0.25	1	0	2.75	
1	0	-0.5	0	1	2.50	
↑						

	x_1	x_2	x_3	x_4	f	b
x_1	1	1.33	-0.33	0	0	1.67
x_4	0	-0.33	0.33	1	0	2.33
	0	-0.67	-0.33	0	1	1.67

$$x_1 = 1.67 ; x_2 = 0 ; x_3 = 0 ; x_4 = 2.33 ; f = 1.67$$