Programación lineal

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PROGRAMACIÓN LINEAL se formula siguiendo el planteamiento general:

$$\min_{x} z$$

← **Función objetivo**

$$s.t.$$

$$h(x) = 0$$

← **Restricciones de igualdad**

$$g(x) \leq 0$$

← **Restricciones de desigualdad**

$$x_{\min} \leq x \leq x_{\max}$$

← **Límite variables**

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La formulación se debe expresar como:

$$\min_x z = c^T x$$ ← **Función objetivo**

$$s.t.$$

$$A_h x = b_h$$ ← **Restricciones de igualdad**

$$A_g x \geq b_g$$ ← **Restricciones de desigualdad**

$$x_{min} \leq x \leq x_{max}$$ ← **Límite variables**

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Símbolos empleados

a_{ij} = coefficient of variable j (column) in i^{th} constraint equation (row)

A = matrix with elements a_{ij} (termed LHS)

b = vector of constraint RHS values with elements b_i

c_j = objective coefficient for variable j

m = number of constraints

n = number of variables

z = Función objetivo = $\Sigma c_j x_j$

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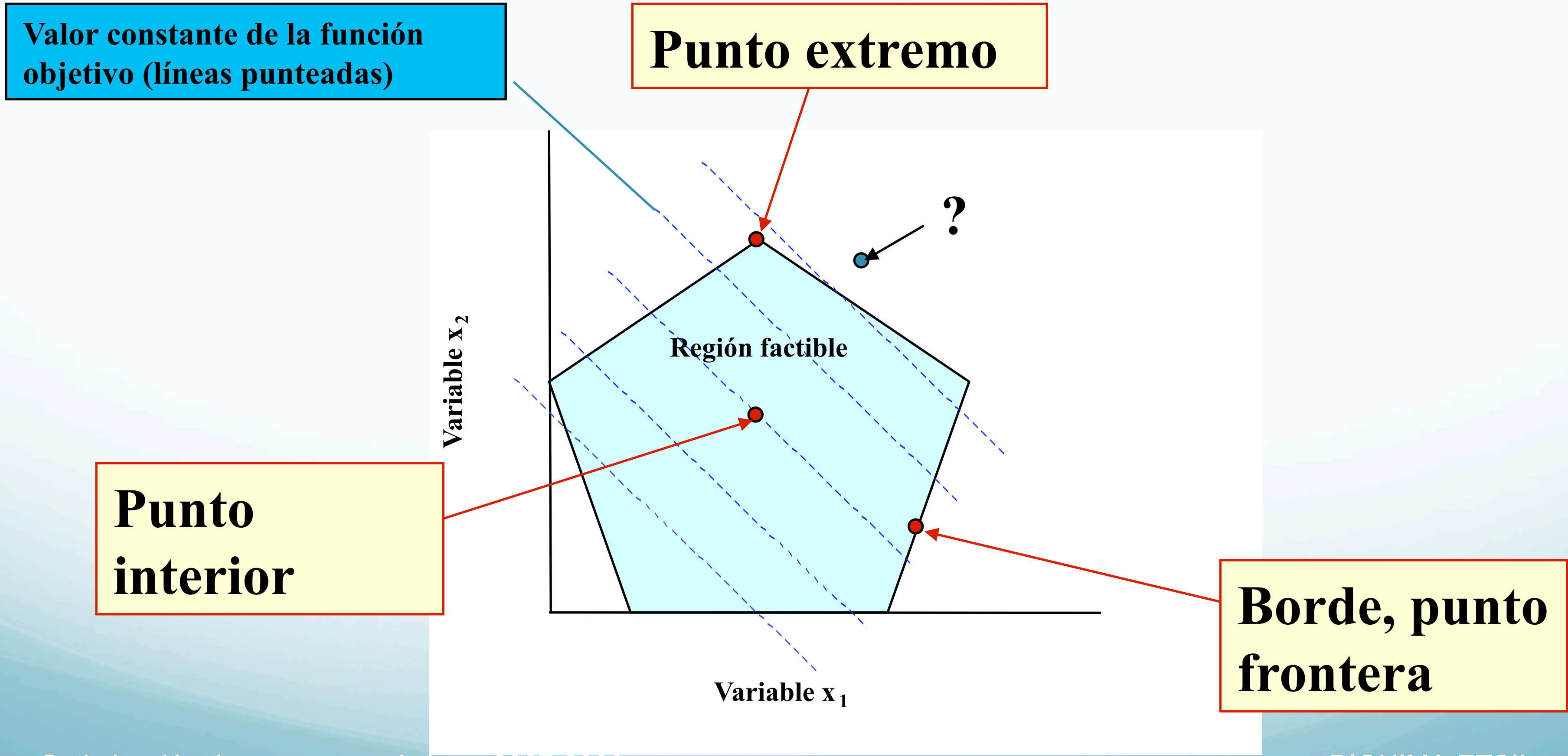
La región factible tiene tres tipos de puntos.



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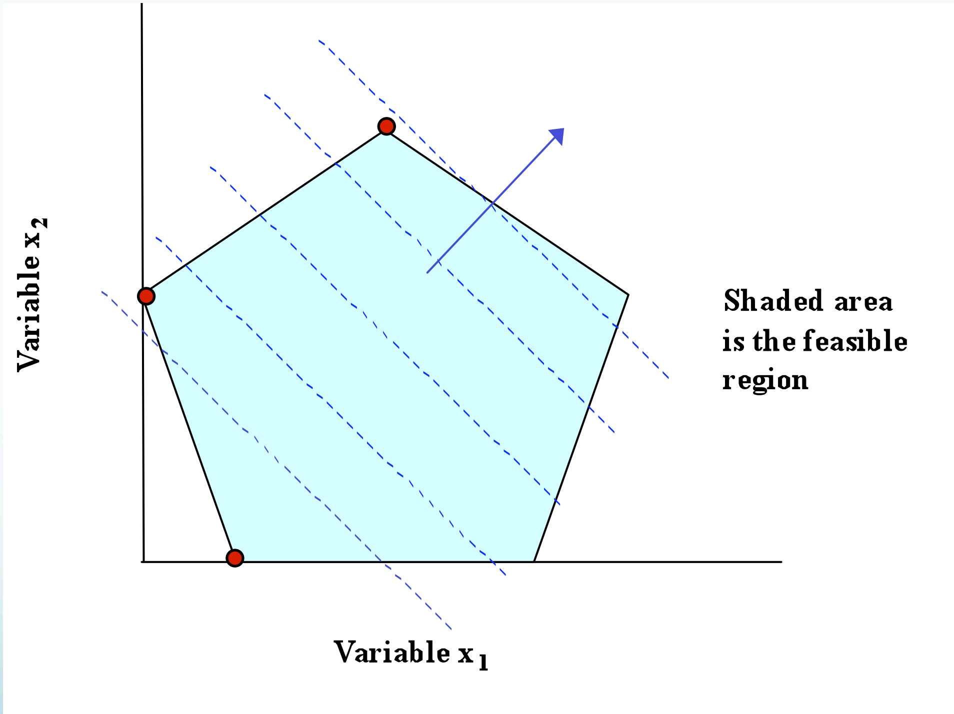
Punto extremo: Un punto es extremo si cualquier segmento de la región factible que contiene al punto tiene a éste al final del segmento. También conocido como vértice.



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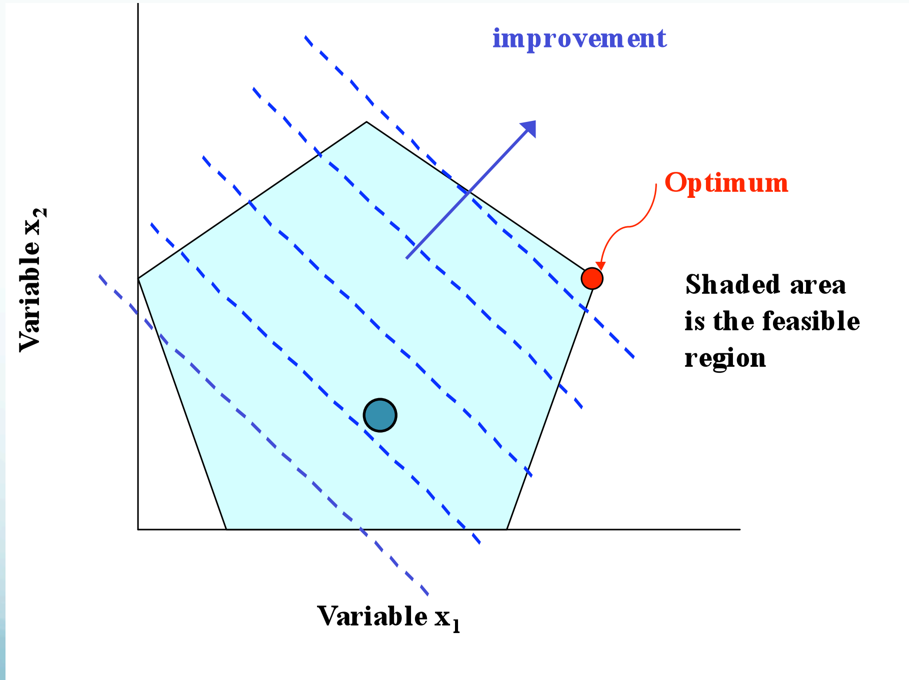
El mejor vértice debe ser el valor óptimo de la función objetivo! Si el problema tiene solución óptima ésta debe de estar en un vértice; si tiene múltiples soluciones óptimas al menos dos deben de estar en punto extremos.



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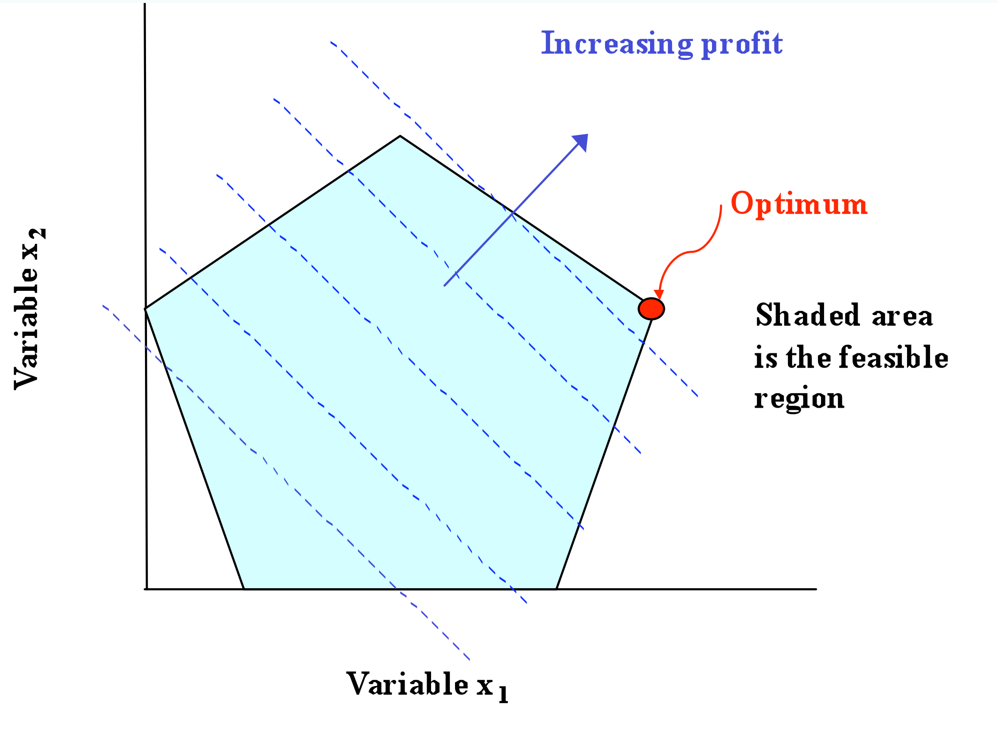
Es un problema de optimización convexa; luego, ¡un óptimo local es un óptimo global!



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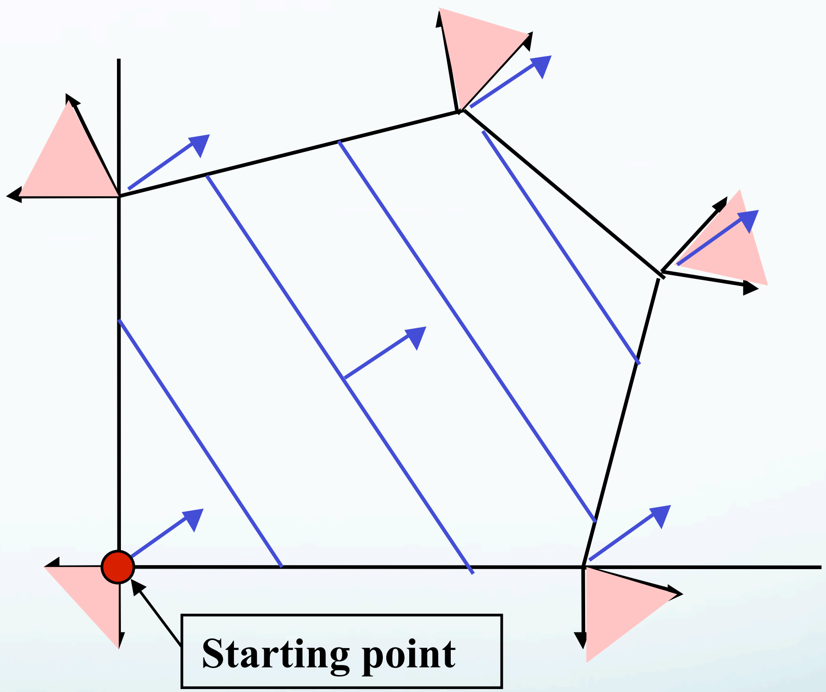
BASIC NUMERICAL OPTIMIZATION

- We only have <u>local information</u> at the "current" point in an iterative scheme.
- We determine a good next point, which will <u>improve the objective value</u>
- We check if further improvement is possible. If yes, continue
- Stop at local optimum.



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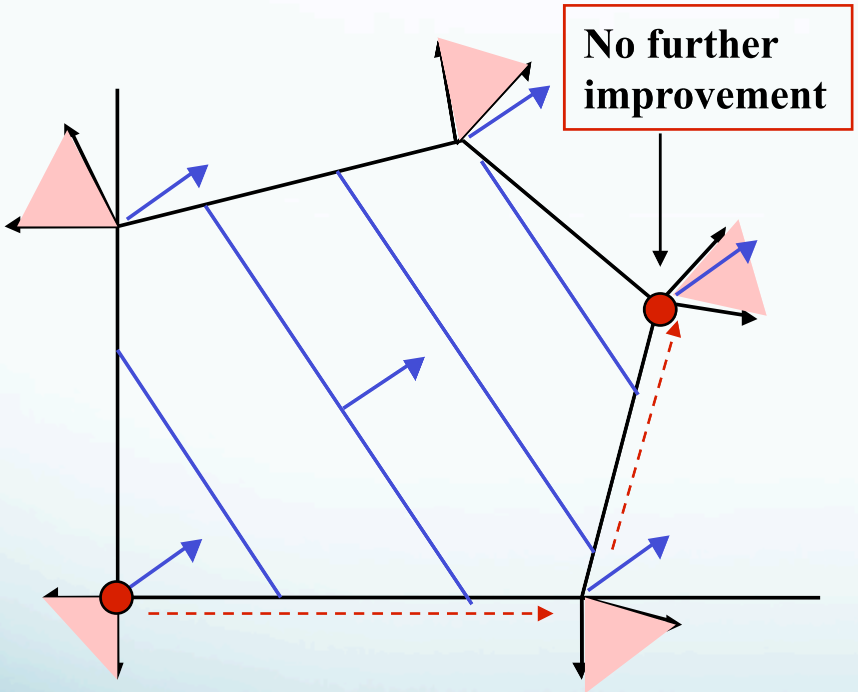
Basis of the Simplex Algorithm



One approach would

- Consider only <u>adjacent</u> corner points for improvement direction.
- Move <u>along the edge</u> that yields the greatest rate of improvement
- Move until another corner point has been reached
- If further improvement is possible, iterate.

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The Simplex algorithm - Using the Corner Point concept as the foundation for an efficient solution method.

We want to convert this general formulation to a system of linear equations - we know how to solve these!

$$\min_x z = c^T x$$

$$s.t.$$

$$A_h x = b_h$$

$$A_1 x \geq b_1$$

$$A_2 x \leq b_2$$

$$x \geq 0$$

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Standard formulation of LPs

$$\min_{x} z = c^T x$$

$$s.t.$$

$$Ax = b$$

$$x \geq 0$$

How can we formulate the problem...

$$\min_x z = c^T x$$
$$s.t.$$
$$Ax = b$$
$$x \geq 0$$

•If we want to maximize a function?

Use the negative of the same function. *Max f = Min -f*

•If we have unconstrained variables, i.e., x may be negative?

x may be written as the difference of two non-negative variables, $x=x_1-x_2$

•If we have inequalities (besides the equalities)?

We can add non-negative additional (slack) variables and convert the inequality into a equality.

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We will add "slack" variables to all inequalities to convert them to equalities.

Original expression

$$5\mathbf{x}_1 + 7\mathbf{x}_2 - 2.3\mathbf{x}_3 \leq 37$$

$$5\mathbf{x}_1 + 7\mathbf{x}_2 - 2.3\mathbf{x}_3 \geq 37$$

$$5\mathbf{x}_1 + 7\mathbf{x}_2 - 2.3\mathbf{x}_3 = 37$$

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The Simplex algorithm - Using the Corner Point concept as the foundation for a solution method.

We have the LP problem in "Standard Form". Note that the system has more variables than equations.

$$\min_{x} z = c^T x$$

$$s.t.$$

$$A\,x = b$$

$$x \geq 0 \qquad \text{[note that ultimately, } x_{min} \geq x \geq x_{max}\text{]}$$

The vector x of variables now includes the slack variables.

Some definitions...

- **Feasible solution.** A solution that satisfies the constraints
- **Basic solution.** A solution in which n-m variables are set equal to zero, and the remaining systems of equations is solved.
- **Basis.** The collection of variables not set equal to zero to obtain the basic solution
- **Basic feasible solution**. This is a basic solution that satisfies the non-negativity conditions.
- **Nondegenerate basic feasible solution.** This is a basic feasible solution that has got exactly m positive x_i.
- **Optimal solution.** A feasible solution that optimizes the Función objetivo.
- **Optimal basic solution.** This is a basic feasible solution for which the Función objetivo is optimal.

Simplex algorithm

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The Simplex Algorithm: Finding a solution for “non-square set of equations

Basic | Non-basic

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ \cdots & & & \cdots & \cdots & \cdots \\ a_{m1} & & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdots \\ b_m \end{bmatrix}$$

Original, non-square equation set of constraints in standard form

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & & \\ a_{m1} & & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdots \\ b_m \end{bmatrix}$$

Square set of equations that can be solved for the basic variables

$$\begin{bmatrix} x_{m+1} \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}$$

Non-basic variables which take the values that optimize the objective. The values will be at an extreme of allowed range.

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The Simplex Algorithm: Finding a solution for “non-square set of equations

BASIS: How do we select the variables to form a basis?

A basis is formed by any selection of variables that yields a set of linearly independent equations. The equations will be linearly independent if

$$\text{Det}\,(A_B) \neq 0$$

with A_B the square coefficient sub-matrix based on the variables selected.

- **The basic variables will have a unique solution for all variables.**
- **The selection of basic variables is not unique. For the system of m equations with n variables (m>n), typically many bases are possible.**

Exercise. Basis Matrix.

$$-x_1+x_2+x_3 \quad = 1$$

$$x_1+x_2 \quad +x_4 \quad = 2$$

$$x_i>=0 \qquad i=1,...,4$$

Solve the system for the following cases:

Basis matrix with x_3 and x_4

$$B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$x_3=1$, $x_4=2$

$x_1=0$, $x_2=0$

Basic solution. Non degenerate

Basis matrix with x_1 and x_4

$$B=\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$x_1=-1$, $x_4=3$

$x_2=0$, $x_3=0$

NOT feasible

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Relating the equations to the graphical interpretation.

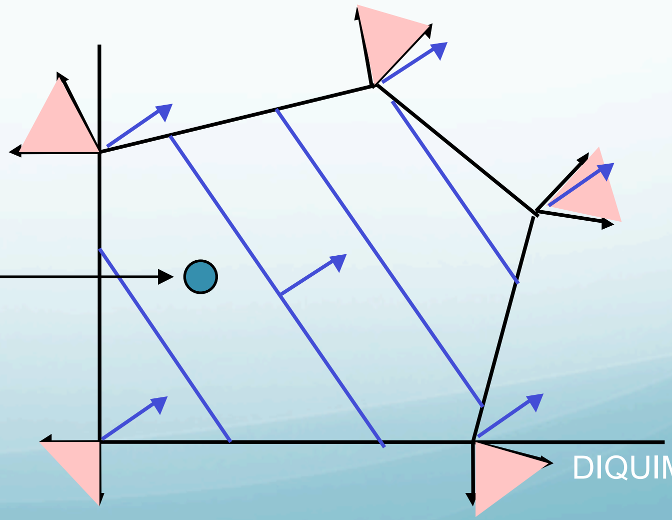
$$\begin{bmatrix} a_{11} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ \cdots & & & \cdots & \cdots & \cdots \\ a_{m1} & & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdots \\ b_m \end{bmatrix} \qquad \begin{bmatrix} x_{m+1} \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} ?? \\ \cdots \\ ?? \end{bmatrix}$$

Square set of equations that can be solved for the basic variables

Non-basic variables which take the values >0

FEASIBLE SOLUTION

Any selection of values for the non-basic variables that result in all variables being non-negative yields a feasible solution.



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Relating the equations to the graphical interpretation.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & & \\ a_{m1} & & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdots \\ b_m \end{bmatrix}$$

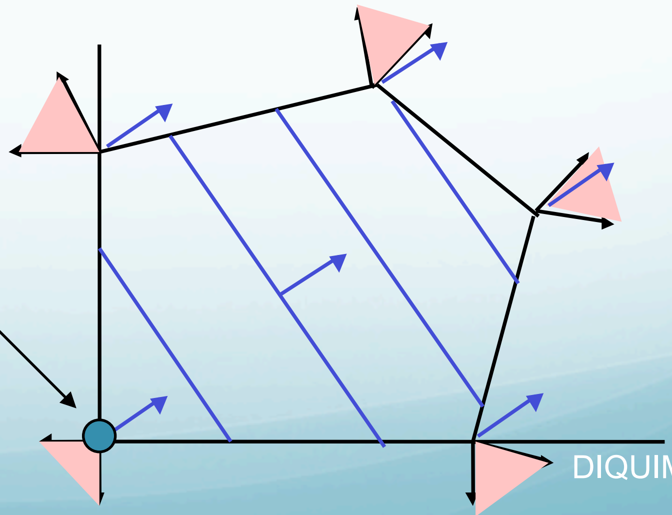
<u>Square</u> set of equations that can be solved for the basic variables

$$\begin{bmatrix} x_{m+1} \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}$$

Non-basic variables which take the values > 0

BASIC FEASIBLE SOLUTION

A selection of 0.0 values for the non-basic variables result in basic variables being non-negative and yields a corner point.



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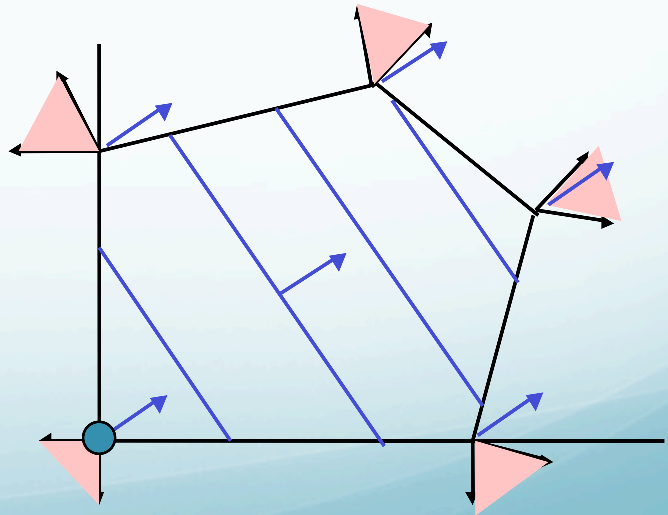
Relating the equations to the graphical interpretation.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & & \\ a_{m1} & & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdots \\ b_m \end{bmatrix} \qquad \begin{bmatrix} x_{m+1} \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}$$

A BASIC FEASIBLE SOLUTION IS A CORNER POINT

- **Any small, negative change in a non-basic variable leads to an infeasibility.**
- **Any point between the solution and a feasible point has the solution at the end point of a line.**
- **Therefore, a BFS is a corner point.**



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The Simplex Algorithm

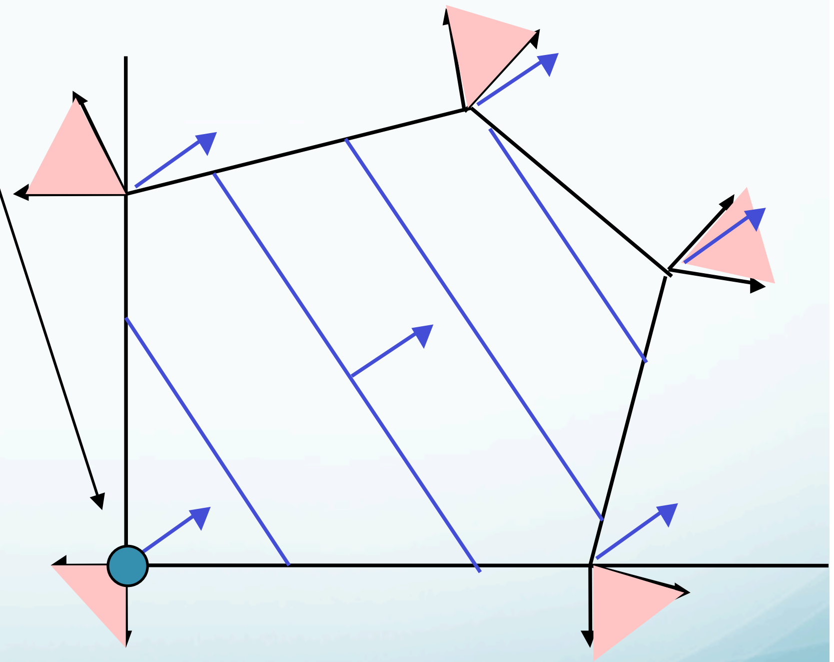
Using the Corner Point concept as the foundation for a solution using the Simplex Algorithm.

Start here

1. **Are we optimal;**

2. **Move along an edge to an adjacent corner point.**

3. **This adds one variable to the basis and removes another variable from the basis.**



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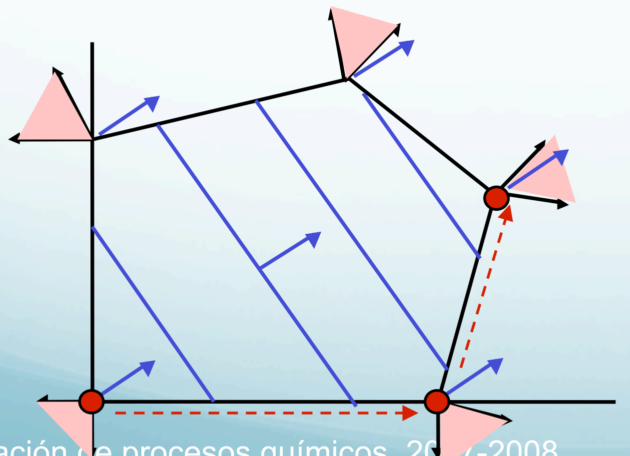
Moving between adjacent corner points is achieved by changing one variable from basic to non-basic and one non-basic to basic. (The new variables must form a basis.)

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & & \\ a_{m1} & & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdots \\ b_m \end{bmatrix} \qquad \begin{bmatrix} x_{m+1} \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}$$



The Simplex Algorithm

- **Consider only adjacent corner points for improvement direction.**
- **Move along the edge that yields the greatest rate of improvement**
- **Move until another corner point has been reached**
- **If further improvement is possible, iterate.**

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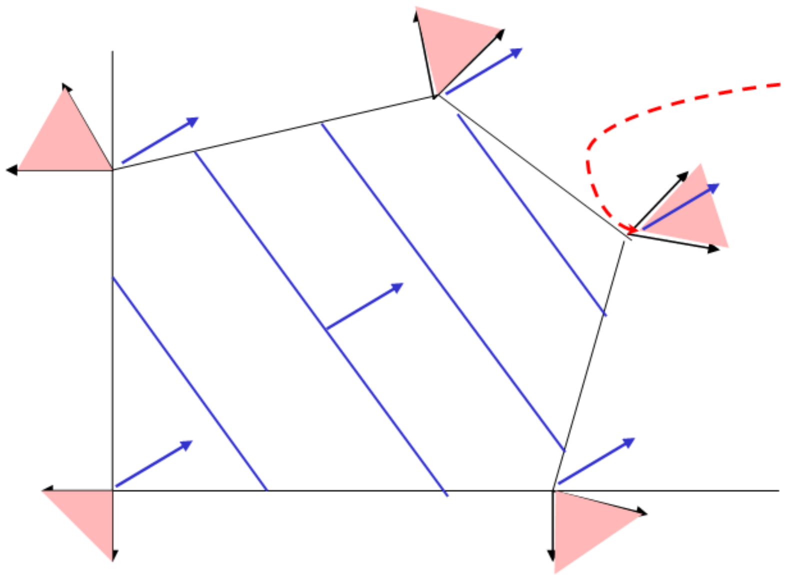
The Simplex Algorithm: Defining the best corner point

Basic | Non-basic

$$\begin{bmatrix} a_{11} & \dots & a_{1m} & a_{1,m+1} & \dots & a_{1n} \\ \dots & & & \dots & \dots & \dots \\ a_{m1} & & a_{mm} & a_{m,m+1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$$

Original, non-square equation set of constraints in standard form



The optimum

The gradient of the objective decrease must lie within the the cone of the gradients of the active constraints.

Each corner point represents a set of basic/ non-basic

Algoritmo Simplex: Procedimiento.

$$\begin{aligned} &\text{Mín:} && f(\underline{x}) = -x_1 + x_2 \\ &\text{Sujeto a:} && 2x_1 - x_2 \geq -2 \\ &&& -x_1 + 3x_2 \geq -2 \\ &&& -x_1 - x_2 \geq -4 \\ &&& x_1, x_2 \geq 0 \end{aligned}$$

Etapa 0. Transformar las restricciones de desigualdad para que todos los bj sean positivos.

$$\begin{aligned} &\text{Mín:} && f(\underline{x}) = -x_1 + x_2 \\ &\text{Sujeto a:} && -2x_1 + x_2 \leq 2 \quad (A) \\ &&& x_1 - 3x_2 \leq 2 \quad (B) \\ &&& x_1 + x_2 \leq 4 \quad (C) \\ &&& x_1, x_2 \geq 0 \end{aligned}$$

Problema original

Etapa Introducir las variables de holgura (slack variables).

$$\begin{aligned} &\text{Mín:} && f(\underline{x}) = -x_1 + x_2 \\ &\text{Sujeto a:} && -2x_1 + x_2 + x_3 = 2 \quad (A) \\ &&& x_1 - 3x_2 + x_4 = 2 \quad (B) \\ &&& x_1 + x_2 + x_5 = 4 \quad (C) \\ &&& x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Problema aumentado

¿Grados de libertad?

Etapa 2. Encontrar una solución que esté en un vértice de la región factible.

2 grados de libertad. 2 variables independientes. *2 variables NO BÁSICAS*

3 variables dependientes. *3 variables BÁSICAS*

$X_1=0$

$x_2=0$

$$f(\underline{x})=0$$
$$x_3=2 \quad (A)$$
$$x_4=2 \quad (B)$$
$$x_5=4 \quad (C)$$

Mín: $f(\underline{x})=-x_1+x_2$

Sujeto a:
$$-2x_1+x_2+x_3=2 \quad (A)$$
$$x_1-3x_2+x_4=2 \quad (B)$$
$$x_1+x_2+x_5=4 \quad (C)$$
$$x_1,x_2,x_3,x_4,x_5\ge 0$$

Etapa 3. Buscar una nueva solución factible que vaya disminuyendo el objetivo.

$$f(\underline{x})-\sum_i c_i x_i=0$$

Eliminar como variable no básica la de mayor valor

$$\text{mín}\left(\frac{b_j}{a_{j1}}\right)\ge 0$$

Escoger como nueva no básica la de menor valor positivo

$$(A)\ :2/-2=-1$$
$$(B)\ :2/1\quad=2$$
$$(C)\ :4/1\quad=4$$

(C) x_4 en este caso

Etapa 4. Transformar las ecuaciones de acuerdo a la nueva distribución de variables.

Variables básicas: x_1, x_3, x_5
Variables no básicas: x_2, x_4

Restricción más desfavorable

$$(B):\ x_1 = 2 + 3x_2 - x_4$$

$$\begin{aligned}
&(A): && -2(2+3x_2-x_4)+x_2+x_3 = 2 \Rightarrow x_3 - 5x_2 + 2x_4 = 6\\
&(B): && x_1 - 3x_2 + x_4 = 2\\
&(C): && (2+3x_2-x_4)+x_2+x_5 = 4 \Rightarrow x_5 + 4x_2 - x_4 = 2\\
& && f+(2+3x_2-x_4)-x_2 = 0 \Rightarrow f + 2x_2 - x_4 = -2
\end{aligned}$$

Se continua el procedimiento hasta….?

Todos los coeficientes de la función objetivo son negativos.

Procedimiento en forma de tabla

	x_1	x_2	x_3	x_4	x_5	f	b	
x_3	-2	1	1	0	0	0	2	
x_4	[1]	-3	0	1	0	0	2	← fila pivote
x_5	1	1	0	0	1	0	4	
	1	-1	0	0	0	1	0	

columna pivote

	x_1	x_2	x_3	x_4	x_5	f	b
x_3	0	-5	1	2	0	0	6
x_1	1	-3	0	1	0	0	2
x_5	0	4	0	-1	1	0	2
	0	2	0	-1	0	1	-2

	x_1	x_2	x_3	x_4	x_5	f	b
x_3	0	0	1	0.75	1.25	0	8.5
x_1	1	0	0	0.25	0.75	0	3.5
x_2	0	1	0	-0.25	0.25	0	0.5
	0	0	0	-0.5	-0.5	1	-3

$x_3 = 8.5$

$x_1 = 3.5$

$x_2 = 0.5$

Forma canónica

$$\begin{aligned}
&\text{Mín:} && f(\underline{x}) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
&\text{Sujeto a:} && a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
& && a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
& && \vdots \\
& && a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \\
& && x_i \geq 0 \ ; \ b_j \geq 0 \quad (i = 1,n \ ; \ j = 1,m)
\end{aligned}$$

Cuando la solución básica factible inicial no es trivial:

Método de las dos fases

Método de la gran M

Método de las dos fases

$$\begin{aligned} &\text{Mín:} && f = x_1 + 2x_2 \\ &\text{Sujeto a:} && 3x_1 + 4x_2 \geq 5 \\ &&& x_1 + x_2 \leq 4 \end{aligned}$$

$$\begin{aligned} &\text{Mín:} && f = x_1 + 2x_2 \\ &\text{Sujeto a:} && 3x_1 + 4x_2 - x_3 = 5 \quad (A) \\ &&& x_1 + x_2 + x_4 = 4 \quad (B) \\ &&& x_i \geq 0 \end{aligned}$$

$$w + \left(\sum_{i=1}^{m} a_{i1}\right)x_1 + \left(\sum_{i=1}^{m} a_{i2}\right)x_2 + \ldots + \left(\sum_{i=1}^{m} a_{in}\right)x_n = \sum_{i=1}^{m} b_i$$

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + .. + a_{1n}x_n & +x_{n+1} & & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + .. + a_{2n}x_n & & +x_{n+2} & & = b_2 \\ \vdots & & \ddots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n & & & +x_{n+m} & = b_m \end{matrix}$$

Fase I

$$\text{Mín:} \quad f = x_1 + 2x_2$$
$$\text{Sujeto a:} \quad 3x_1 + 4x_2 - x_3 + x_5 = 5 \quad (A)$$
$$x_1 + x_2 + x_4 + x_6 = 4 \quad (B)$$
$$x_i \geq 0$$

$$w + 4x_1 + 5x_2 - x_3 + x_4 = 9$$

x_1	x_2	x_3	x_4	x_5	x_6	w	b	
3	4	-1	0	1	0	0	5	←
1	1	0	1	0	1	0	4	
4	5	-1	1	0	0	1	9	

x_1	x_2	x_3	x_4	x_5	x_6	w	b	
0.75	1	-0.25	0	0.25	0	0	1.25	
0.25	0	0.25	1	-0.25	1	0	2.75	←
0.25	0	0.25	1	-1.25	0	1	2.75	

x_1	x_2	x_3	x_4	x_5	x_6	w	b
1	1	0	1	0	1	0	4
1	0	1	4	-1	4	0	11
0	0	0	0	-1	-1	1	0

Fase II

Mín: $f = x_1 + 2x_2$

Sujeto a: $3x_1 + 4x_2 - x_3 = 5 \quad (A)$

$x_1 + x_2 + x_4 = 4 \quad (B)$

$x_i \geq 0$

$x_2 = 4 \ ; \ x_3 = 11.$

$$f = x_1 + 2(4 - x_1 - x_4) \Rightarrow f + x_1 + 2x_4 = 8$$

x_1	x_2	x_3	x_4	f	b	
1	1	0	1	0	4	
1	0	1	4	0	11	←
1	0	0	2	1	8	
			↑			

x_1	x_2	x_3	x_4	f	b	
0.75	1	-0.25	0	0	1.25	←
0.25	0	0.25	1	0	2.75	
1	0	-0.5	0	1	2.50	
↑						

	x_1	x_2	x_3	x_4	f	b
x_1	1	1.33	-0.33	0	0	1.67
x_4	0	-0.33	0.33	1	0	2.33
	0	-0.67	-0.33	0	1	1.67

$$x_1 = 1.67 \ ; \ x_2 = 0 \ ; \ x_3 = 0 \ ; \ x_4 = 2.33 \ ; \ f = 1.67$$